

SESHADRI CONSTANTS OF $K3$ SURFACES OF DEGREES 6 AND 8

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ABSTRACT. We compute Seshadri constants $\varepsilon(X) := \varepsilon(\mathcal{O}_X(1))$ on $K3$ surfaces X of degrees 6 and 8. Moreover, more generally, we prove that if X is any embedded $K3$ surface of degree $2r - 2 \geq 8$ in \mathbb{P}^r not containing lines, then $1 < \varepsilon(X) < 2$ if and only if the homogeneous ideal of X is not generated by only quadrics (in which case $\varepsilon(X) = \frac{3}{2}$).

1. INTRODUCTION AND RESULTS

In the past couple of decades there has been considerable interest in studying the local positivity of nef line bundles on algebraic varieties. Demailly [De] introduced Seshadri constants to capture the concept of local positivity. Let X be a smooth projective variety and L a nef line bundle on X . For a point $x \in X$ the real number

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\text{mult}_x C}$$

is the *Seshadri constant of L at x* . Here the infimum is taken over all curves C passing through x and it is easily seen that one can restrict to considering only irreducible curves. Equivalently,

$$\varepsilon(L, x) := \sup\{\varepsilon \in \mathbb{R} \mid f^*L - \varepsilon E \text{ is nef}\},$$

where $f : \tilde{X} \rightarrow X$ is the blow up at x and $E \subset \tilde{X}$ is the exceptional divisor.

The (*global*) *Seshadri constant of L* is defined as

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x).$$

One has $\varepsilon(L) > 0$ if and only if L is ample, by Seshadri's criterion. Moreover, $\varepsilon(L) \leq \sqrt[n]{L^n}$, where $n := \dim X$, by Kleiman's theorem [E-K-L, Rem. 1.8].

Using standard terminology, we say that a curve C is a *Seshadri curve (of L)* if C computes $\varepsilon(L)$, that is, if $\varepsilon(L) = \frac{L \cdot C}{\text{mult}_x C}$ for a point $x \in C$.

We refer to [Ba2, La, B⁺⁺], to mention a few, for accounts on Seshadri constants and the development in the research on them.

One of the very subtle points about Seshadri constants is that their values are only known in few cases. For instance, all known examples are rational and it is not even known whether Seshadri constants are always rational or not.

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If X is a $K3$ surface and L is ample, then the following is known:

- If L is not globally generated, then $\varepsilon(L) = \frac{1}{2}$. Otherwise, $\varepsilon(L) \geq 1$ [B-DR-S, Prop. 3.1]. (Indeed, by the nowadays classical results of Saint-Donat [SD], an ample line bundle L on a $K3$ is not globally generated if and only if there is an elliptic pencil $|E|$ such that $E \cdot L = 1$. In this case, the Seshadri curves are the curves in $|E|$ with a double point.)
- If L is globally generated but not very ample, then $\varepsilon(L) = 1$ or 2 . This also follows from [SD] and is probably well-known to the experts, but we give a proof in Proposition 3.1 in § 3 for lack of a reference.
- If L is very ample, then $\varepsilon(L) = 1$ if and only if X embedded by $|L|$ contains a line [Ba2, Thm. 2.1(a)]. (This holds on any surface.) Moreover, it is well-known that $K3$ surfaces containing a line form a codimension one subset in the moduli space of polarized $K3$ surfaces of fixed degree.
- If L is very ample with $L^2 = 4$, that is, X embedded by $|L|$ is a smooth quartic surface in \mathbb{P}^3 , then $\varepsilon(L) = 1, \frac{4}{3}$ or 2 and all three cases occur [Ba1]: $\varepsilon(L) = 1$ if and only if X contains a line; $\varepsilon(L) = \frac{4}{3}$ if and only if X contains no line and contains a rational curve $C \in |L|$ with a triple point (which happens if and only if X contains a point where the Hesse form vanishes); $\varepsilon(L) = 2$ in all other cases. Furthermore, the two first cases occur on codimension one subsets in the space of quartic surfaces.
- If $\text{Pic } X \cong \mathbb{Z}[L]$ and L^2 is a square, then $\varepsilon(L) = \sqrt{L^2}$ [Kn1].

By the above, when studying Seshadri constants on a $K3$ surface X , one can without loss of generality assume that L is very ample, whence that the complete linear system $|L|$ embeds X as a surface of degree $L^2 = 2r - 2$ in \mathbb{P}^r , where $r := \dim |L|$. We set $\varepsilon(L) = \varepsilon(X)$. Moreover, we can also assume that X does not contain any lines, that is, that $\varepsilon(X) > 1$.

In this note we will compute the Seshadri constant $\varepsilon(X)$ of X if $L^2 = 6$ or 8 . It is well-known that the homogeneous ideal of a $K3$ surface X of degree $2r - 2$ in \mathbb{P}^r is always generated by quadrics and cubics [SD]. In particular, X is a complete intersection of type $(2, 3)$ in \mathbb{P}^4 , that is, a complete intersection of a quadric and a cubic hypersurface if $L^2 = 6$. By contrast, if $L^2 = 8$, there are two types of projective models. In general X is a complete intersection of type $(2, 2, 2)$ in \mathbb{P}^5 , that is, of three hyperquadrics. In addition, there is a codimension one subspace of polarized $K3$ surfaces of degree 8 that are a section of $|\mathcal{O}_Y(3) - \mathcal{F}|$ in a smooth three-dimensional rational normal scroll Y in \mathbb{P}^5 , where \mathcal{F} denotes the class of the \mathbb{P}^2 -fibers [SD, J-K]. The homogeneous ideal of these cannot be generated only by quadrics. In this latter case, $\varepsilon(L) = \frac{3}{2}$, by the following more general result, which we will prove in § 3.

Proposition 1.1. *Let $X \subset \mathbb{P}^r$ be a $K3$ surface of degree $2r - 2$ not containing lines, $r \geq 5$. Then $1 < \varepsilon(X) < 2$ if and only if the homogeneous ideal of X is not generated only by quadrics, in which case $\varepsilon(X) = \frac{3}{2}$ and the irreducible Seshadri curves are the irreducible curves with a double point in an elliptic pencil of degree 3.*

Now to our main result, whose proof is also contained in § 3.

Theorem 1.2. *Let X be a $K3$ surface not containing any lines.*

(a) If $X \subset \mathbb{P}^4$ is a surface of type $(2,3)$, then one of the following cases occurs:

- (a-i) $\varepsilon(X) = 2$, and there is a Seshadri curve that is a hyperplane section with a triple point;
- (a-ii) $\varepsilon(X) = \frac{3}{2}$, and all irreducible Seshadri curves are irreducible curves with a double point in an elliptic pencil of degree 3;

(b) If $X \subset \mathbb{P}^5$ is a surface of type $(2,2,2)$, then one of the following cases occurs:

- (b-i) $\varepsilon(X) = \frac{8}{3}$, and there is a Seshadri curve that is a hyperplane section with a triple point;
- (b-ii) $\varepsilon(X) = \frac{5}{2}$, and all irreducible Seshadri curves are irreducible curves with a double point in an elliptic pencil of degree 5;
- (b-iii) $\varepsilon(X) = 2$, and all irreducible Seshadri curves are either conics or irreducible curves with a double point in an elliptic pencil of degree 4.

Furthermore, cases (a-ii), (b-ii) and (b-iii) occur on sets of codimension one in the space of all such surfaces.

The proof is based on existence results of curves with triple points on a $K3$ surface in [Ga] and in Section 2 of this paper, combined with properties of curves on $K3$ surfaces. In § 4 we make some comments on how far similar reasonings for curves with singularities of higher multiplicities would bring us.

Conventions. We work over the field of complex numbers. We will assume familiarity with standard results on $K3$ surfaces, like the results in [SD] (a brief summary of those can be found in [Kn2, § 2]), and about deformations and families of $K3$ surfaces, as explained for instance in [B-P-H-V, G-H2, Ko].

2. CURVES WITH A TRIPLE POINT ON A $K3$ SURFACE OF DEGREE 6

The main ingredient in Theorem 1.2 is the existence of irreducible curves in the linear system $|\mathcal{O}_X(1)|$ with a singularity of multiplicity 3 at a (special) point of X , where X is a general $K3$ surface of degree $2r - 2$ in \mathbb{P}^r , with $r = 4, 5$. For $r = 5$ this follows by [Ga, Cor. 4.3]. The argument used in [Ga, Section 4] does not apply to the case $r = 4$ (see [Ga, Rmk. 4.4]). The existence of elliptic curves in $|\mathcal{O}_X(1)|$ with a triple point will be proved in this section. We need to recall some classical deformation theory of $K3$ surfaces (cf. [C-L-M]).

Let R_1 and R_2 , with $R_1 \simeq R_2 \simeq \mathbb{F}_1$, be two general rational normal scrolls of degree 3 in \mathbb{P}^4 , generated by the union of the secants to divisors in two general g_2^1 's on an elliptic normal curve $E \subset \mathbb{P}^4$ of degree 5. Then $E = R_1 \cap R_2$ and the intersection is transverse. Moreover, if σ_i and F_i are the two generators of $\text{Pic } R_i$, with $\sigma_i^2 = -1$ and $F_i^2 = 0$, we have that $|\mathcal{O}_{R_i}(1)| = |\mathcal{O}_{R_i}(\sigma_i + 2F_i)|$, for $i = 1, 2$. By classical deformation theory of $K3$ surfaces (cf. [C-L-M, Cor. 1, Thms. 1 and 2] and related references, precisely, [Fr, Rmk. 2.6] and [G-H1, §2]), we know that, no matter how we choose 16 general points $\{\xi_1, \dots, \xi_{16}\}$ on E , there exists a smooth family of surfaces $\mathcal{X} \rightarrow \mathbb{A}^1$ whose general fibre \mathcal{X}_t is a smooth $K3$ surface of degree 6 in \mathbb{P}^4 such that $\text{Pic}(\mathcal{X}_t) \simeq \mathbb{Z}[\mathcal{O}_{\mathcal{X}_t}(1)]$ and whose special fibre is $\mathcal{X}_0 = R_1 \cup \tilde{R}_2$, where \tilde{R}_2 is the blowing-up of R_2 at $\{\xi_1, \dots, \xi_{16}\}$

with exceptional divisors E_1, \dots, E_{16} . We want to obtain curves with a triple point in $|\mathcal{O}_{\mathcal{X}_t}(1)|$ as deformations of suitable curves in $|\mathcal{O}_{\mathcal{X}_0}(1)|$.

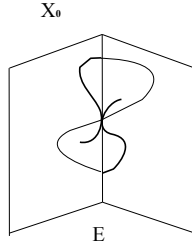
Lemma 2.1. *Let $R = R_1 \cup R_2 \subset \mathbb{P}^4$ be the union of two general rational normal scrolls as above. Then there exists a unique curve $C = C_1 \cup C_2 \in |\mathcal{O}_R(1)|$, where $C_i \subset R_i$, $i = 1, 2$, such that both C_i 's have a node at the same point of E with one branch tangent to E .*

Proof. Assume that there exists a curve $C = C_1 \cup C_2 \in |\mathcal{O}_R(1)|$ as in the statement. Let $p \in E$ be the point where both C_i 's have a node with one branch tangent to E . Then $C_i \subset R_i$ intersects the unique divisor D_i in $|F_i|$ passing through p with multiplicity 2. As $F_i \cdot C_i = F_i \cdot (\sigma_i + 2F_i) = 1$, we have $D_i \subset C_i$. More precisely, we have

- (1) $C_i = D_i \cup L_i$, where $D_i \sim F_i$ and $L_i \sim \sigma_i + F_i$, for $i = 1, 2$;
- (2) $C \cap E = 3p + q_1 + q_2$, where p, q_1, q_2 are distinct points satisfying

$$\begin{aligned} D_1 \cap E \text{ (resp. } D_2 \cap E) &= p + q_2 \text{ (resp. } p + q_1) \text{ and} \\ L_1 \cap E \text{ (resp. } L_2 \cap E) &= 2p + q_1 \text{ (resp. } 2p + q_2). \end{aligned}$$

In particular, C has nodes at q_1 and q_2 , a space quadruple point at p as in the following figure and it is smooth on $R \setminus E$. It follows that the existence of a



curve C as in the statement is equivalent to the existence of a unique point $p \in E$ such that the unique divisor $L_1 \sim \sigma_1 + F_1$ on R_1 tangent to E at p satisfies $L_1 \cap E = 2p + q_1$, where $q_1 \neq p$ and $p + q_1 \sim F_2$ on E . We claim that this is satisfied by the point p that is the unique member in the degree one linear system $|\mathcal{O}_E(\sigma_1 + F_1 - F_2)|$ on E . Indeed, if $D_2 \sim F_2$ is the unique divisor on R_2 passing through p , we may assume $D_2 \cap E = p + q_1$ with $p \neq q_1$, by the generality of the linear system $|\mathcal{O}_E(F_2)|$ on E (spanning the normal scroll R_2 in \mathbb{P}^4). The unique divisor $L_1 \in |\mathcal{O}_{R_1}(\sigma_1 + F_1)|$ cutting $2p + q_1$ on E must be smooth by a trivial argument. Let $D_1 \in |\mathcal{O}_{R_1}(F_1)|$ be the unique divisor passing through p and let $p + q_2 = D_1 \cap E$, where q_2 is different from p and q_1 , because $D_1 \cdot L_1 = 1$. Since $D_1 + L_1 \in |\mathcal{O}_{R_1}(1)|$, we have that $3p + q_1 + q_2 \in |\mathcal{O}_E(1)|$ and thus there exists a unique smooth curve $L_2 \sim \sigma_2 + F_2$ on R_2 cutting $2p + q_2$ on E . This proves our claim, whence also the lemma. \square

Proposition 2.2. *Let X be a general K3 surface of degree 6 in \mathbb{P}^4 such that $\text{Pic}(X) \simeq \mathbb{Z}[\mathcal{O}_X(1)]$. Then in the linear system $|\mathcal{O}_X(1)|$ there exist finitely many elliptic curves with a singularity of multiplicity 3.*

Proof. Let R_1 and R_2 and $\mathcal{X} \rightarrow \mathbb{A}^1$ as above. In the linear system $|\mathcal{O}_{\mathcal{X}_t}(1)|$ on the general fibre \mathcal{X}_t of $\mathcal{X} \rightarrow \mathbb{A}^1$ there are no curves with a singularity of multiplicity bigger than 3, because every such curve must be reducible and $\text{Pic } \mathcal{X}_t \cong \mathbb{Z}[\mathcal{O}_{\mathcal{X}_t}(1)]$. Moreover, if there exist curves with a triple point, then they are finitely many by [Ch, Lemma 3.1]. We want to prove their existence. By Lemma 2.1, there exists a unique curve $C = C_1 \cup C_2 \in |\mathcal{O}_{\mathcal{X}_0}(1)|$ with a space quadruple point at a point $p \in E$ as in the figure on the previous page, and C must satisfy conditions (1) and (2). We will show that C may be deformed to a curve on \mathcal{X}_t in such a way that the non-planar quadruple point of C at p is deformed to a triple point. The scheme parametrizing deformations of C in \mathcal{X} is an irreducible component \mathcal{H} of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbb{A}^1}$ of the family \mathcal{X} . The scheme parametrizing deformations of the singularity of C at p is the versal deformation space $T_{C,p}^1$. By [Ga, Lemma 3.2], we may choose local coordinates x, y, z, t of \mathcal{X} centered at p such that C is given by the equations

$$(1) \quad \begin{cases} (y + x + z^2)z = 0, \\ xy = t, \\ t = 0 \end{cases}$$

at p . Moreover, by [Ga, (10) in proof of Lemma 3.2], the versal deformation family $\mathcal{C}_p \rightarrow T_{C,p}^1$ is given by the equations

$$(2) \quad \begin{cases} (y + x + z^2)z + a_1 + a_2x + a_3y + a_4z = 0, \\ xy + b_1 + b_2z + b_3z^2 = 0, \end{cases}$$

where $(a_1, \dots, a_4, b_1, b_2, b_3) = (\underline{a}, \underline{b})$ are affine coordinates on $T_{C,p}^1$. In particular, we have that $T_{C,p}^1 \simeq \mathbb{C}^7$. Denote by $\mathcal{D} \rightarrow \mathcal{H}$ the universal family parametrized by \mathcal{H} . By versality, there exist analytic neighborhoods U_p of $[C]$ in \mathcal{H} , U'_p of p in \mathcal{D} and V_p of $\underline{0}$ in $T_{C,p}^1$ and a map $\phi_p : U_p \rightarrow V_p$ so that the family $\mathcal{D}|_{U_p} \cap U'_p$ is isomorphic to the pull-back of $\mathcal{C}_p|_{V_p}$, with respect to ϕ_p ,

$$(3) \quad \begin{array}{ccccccc} \mathcal{C}_p & \longleftarrow & \mathcal{C}_p|_{V_p} & \longleftarrow & U_p \times_{V_p} \mathcal{C}_p|_{V_p} & \xrightarrow{\sim} & \mathcal{D}|_{U_p} \cap U'_p \hookrightarrow \mathcal{D} \\ \downarrow & & \downarrow & & \searrow & & \downarrow \\ T_{C,p}^1 & \longleftarrow & V_p & \xleftarrow{\phi_p} & U_p & \hookrightarrow & \mathcal{H}. \end{array}$$

We need to describe the image $\phi_p(U_p) \subset V_p$. First observe that the map ϕ_p is finite. To see this, observe that the fibre $\phi_p^{-1}(\underline{0})$ over $\underline{0}$ is the analytically equisingular deformation locus of $[C]$ in U_p . Since C has a space singularity, all equisingular deformations of C are trivially contained in \mathcal{X}_0 . By the previous lemma, we deduce that $\phi_p^{-1}(\underline{0})$ consists of the unique point $[C]$, possibly with multiplicity. Thus $\phi_p(U_p) \subset T_{C,p}^1$ is an irreducible analytic subvariety of dimension $5 = \dim |\mathcal{O}_{\mathcal{X}}(1)| = \dim |\mathcal{O}_{\mathbb{P}^4}(1)| + 1$, containing $\phi_p(U_p \cap |\mathcal{O}_{\mathcal{X}_0}(1)|)$ as a codimension 1 subvariety. Moreover, using that the dimension of $\phi_p(U_p \cap |\mathcal{O}_{\mathcal{X}_0}(1)|)$ is 4 and that it parametrizes all deformations of the curve corresponding to $\phi_p([C])$ on the surface $xy = 0$, we have that $\phi_p(U_p \cap |\mathcal{O}_{\mathcal{X}_0}(1)|)$ is given by the equations $b_1 = b_2 = b_3 = 0$. We denote by \mathcal{T} the subvariety of $T_{C,p}^1$ defined as the Zariski closure of the locus of points $(\underline{a}, \underline{b})$ that correspond to curves

with an ordinary singularity of multiplicity 3 at a point $(x_0, y_0, z_0) \neq (0, 0, z_0)$ of a smooth surface $xy + b_1 + b_2z + b_3z^2 = 0$. By [Ga, proof of Thm. 3.9] we know that \mathcal{T} is nonempty and intersects the locus $b_2 = b_3 = 0$ along the curve $\gamma : a_1 = a_2 = a_3 = b_2 = b_3 = 4b_1 - a_4^2 = 0$. In particular, we have that $\phi_p(U_p \cap |\mathcal{O}_{\mathcal{X}_0}(1)|) \cap \mathcal{T} = \emptyset$. It follows that $\dim(\phi_p(U_p) \cap \mathcal{T}) \leq 1$, and we claim that equality is attained.

To see this, we first note that every irreducible component of \mathcal{T} has codimension at most 4 in $T_{C,p}^1$. (Indeed, using that $(x_0, y_0) \neq (0, 0)$, we can recover x or y from the second equation in (2). Substituting into the first, we get the equation of a planar curve. Imposing that this curve has a triple point, one obtains four equations defining \mathcal{T} .) It follows that $\dim(\phi_p(U_p) \cap \mathcal{T}) \geq 5 + 3 - 7 = 1$, proving that $\dim(\phi_p(U_p) \cap \mathcal{T}) = 1$, as desired.

This finishes the proof of the proposition, by versality. \square

3. COMPUTING SESHADRI CONSTANTS

We start with the following result mentioned in the introduction:

Proposition 3.1. *Let L be an ample, globally generated line bundle on a K3 surface X . Then L is not very ample if and only if one of the following occurs:*

- (a) $L^2 = 2$;
- (b) *there is an elliptic pencil $|E|$ such that $E \cdot L = 2$;*
- (c) $L \sim 2B$, *with B globally generated such that $B^2 = 2$.*

Furthermore, we have $\varepsilon(L) = 1$ in cases (a)-(b) and $\varepsilon(L) = 2$ in case (c).

Proof. The characterization of L as in (a)-(c) follows from [SD]. Moreover, $\varepsilon(L) \geq 1$ as L is globally generated and ample [B-DR-S, Prop. 3.1].

It is well-known that any elliptic pencil on a K3 surface contains singular members. Thus $\varepsilon(L) = 1$ in case (b).

In case (a), the linear system $|L|$ defines a double cover of X onto \mathbb{P}^2 branched along a reduced plane sextic. The pullback C of any line tangent to the sextic has a point of multiplicity two, whence $\varepsilon(L) = 1$. It also follows that $\varepsilon(L) = 2\varepsilon(B) = 2$ in case (c). \square

We recall the following fact: If $\varepsilon(L) < \sqrt{L^2}$, there is an irreducible Seshadri curve C on S , that is, an irreducible curve C and a point $x \in C$ such that $\varepsilon(L) = \frac{L \cdot C}{\text{mult}_x C}$, cf. [Og, Lemma 2.1] or [Sz, Lemma 3.1]. For any curve C we set

$$\varepsilon_{C,x} := \frac{L \cdot C}{\text{mult}_x C} \quad \text{and} \quad m_x := \text{mult}_x C.$$

If C is irreducible, then, as $p_a(C) - p_g(C) \geq \frac{1}{2}m_x(m_x - 1)$, we obtain by adjunction on a K3 surface that

$$(4) \quad C^2 \geq m_x(m_x - 1) - 2.$$

We will repeatedly make use of the following well-known fact: Any linear system $|D|$ on a K3 surface satisfying $D^2 \geq -2$ and $D \cdot H > 0$ for some ample divisor H is nonempty. If in addition $D^2 \geq 0$, then $|D|$ contains a singular member, that is, an element (possibly nonreduced or reducible) with a point of multiplicity ≥ 2 .

Proof of Proposition 1.1. Assume that $1 < \varepsilon(X) < 2$ and set $L := \mathcal{O}_X(1)$. As $2 < \sqrt{L^2}$, there exists an irreducible Seshadri curve C with $x \in C$ such that $\varepsilon_{C,x} = \varepsilon(X)$. Then $2 \leq m_x < L.C < 2m_x$. Hence, by (4) and the Hodge index theorem,

$$L^2(m_x(m_x - 1) - 2) \leq L^2 C^2 \leq (L.C)^2 < 4m_x^2.$$

Using the facts that $L^2 \geq 8$ and $m_x \geq 2$, one easily verifies that the only possibility is $(m_x, C^2, L.C) = (2, 0, 3)$, whence $\varepsilon(X) = \frac{3}{2}$. Furthermore, the existence of C implies that the homogeneous ideal of X is not generated only by quadrics by [SD].

Conversely, assume that the homogeneous ideal of X is not generated only by quadrics. Then by [SD] there exists an elliptic pencil $|E|$ such that $E.L = 3$. As above, any member $C \in |E|$ with a point x of multiplicity two satisfies $\varepsilon_{C,x} = \frac{3}{2}$, whence $\varepsilon(L) \leq \frac{3}{2}$. Furthermore, $\varepsilon(L) > 1$ as X does not contain lines. \square

Proof of Theorem 1.2. We set $L := \mathcal{O}_X(1)$.

We first treat the case $X = (2, 3) \subset \mathbb{P}^4$, where $L^2 = 6$.

By Proposition 2.2, on the general polarized $K3$ surface (X, L) of degree 6 there is an irreducible curve $C \in |L|$ with a triple point x . Hence $\varepsilon(L) \leq \varepsilon_{C,x} = \frac{6}{3} = 2$. By lower semi-continuity of Seshadri constants in a family of smooth surfaces [Og, Cor. 5], we have $\varepsilon(L) \leq 2$ for any (X, L) . Furthermore, if equality holds, there is a hyperplane section with a triple point that is a Seshadri curve.

Assume now that $\varepsilon(L) < 2$. As $2 < \sqrt{6}$, there must exist an irreducible Seshadri curve C with $x \in C$ such that $\varepsilon_{C,x} = \varepsilon(L) < 2$. From (4) and the Hodge index theorem, we have

$$(5) \quad 6(m_x(m_x - 1) - 2) \leq 6C^2 \leq (L.C)^2 < 4m_x^2.$$

If $m_x \geq 3$, this can only be satisfied if $(m_x, L.C, C^2) = (3, 5, 4)$. But then $D := L - C$ satisfies $D^2 = 0$ and $L.D = 1$, contradicting the fact that L is globally generated, by [SD].

Since $m_x \geq 2$, as X is assumed not to contain lines, we are left with the case $m_x = 2$, where (5) yields $C^2 = 0$ and $L.C = 3$ (using [SD] and the fact that L is very ample). This means that C is a singular member of an elliptic pencil. By e.g. [Kn2, Thm. 1.1] there exist $K3$ surfaces of degree 6 with the Picard group generated by the hyperplane section and an elliptic degree 3 curve. This occurs on sets of codimension one in the space of all such surfaces, cf. [Ko, Thm. 14] or [G-H2, p. 594]. Hence the theorem follows in this case.

We next treat the case $X = (2, 2, 2) \subset \mathbb{P}^5$, where $L^2 = 8$.

By [Ga, Thm. 1.1], on the general polarized $K3$ surface (X, L) of degree 8 there is an irreducible curve $C \in |L|$ with a triple point x . Hence $\varepsilon(L) \leq \varepsilon_{C,x} = \frac{8}{3}$. By lower semi-continuity again, $\varepsilon(L) \leq \frac{8}{3}$ for any (X, L) , and, if equality holds, there is a hyperplane section with a triple point that is a Seshadri curve.

Assume now that $\varepsilon(L) < \frac{8}{3}$. As $\frac{8}{3} < \sqrt{8}$, there must exist an irreducible Seshadri curve C with $x \in C$ such that $\varepsilon_{C,x} = \varepsilon(L) < \frac{8}{3}$. From (4) and the Hodge index theorem,

$$(6) \quad 8(m_x(m_x - 1) - 2) \leq 8C^2 \leq (L.C)^2 < \frac{64}{9}m_x^2.$$

One easily checks that this cannot be satisfied for $m_x \geq 9$.

If $m_x = 8$, the only solution to (6) is $(L.C, C^2) = (21, 54)$. Set $D := 3L - C$. Then $(D^2, L.D) = (0, 3)$, so any curve $D_0 \in |D|$ with a singular point y satisfies $\varepsilon_{D_0, y} \leq \frac{3}{2} < \frac{21}{8} = \varepsilon_{C, x} = \varepsilon(L)$, a contradiction.

If $m_x = 7$, the only solution to (6) is $(L.C, C^2) = (18, 40)$. Set $D := C - 2L$. Then $(D^2, L.D) = (0, 2)$, contradicting the very ampleness of L by [SD].

If $m_x = 6$, the only solution to (6) is $(L.C, C^2) = (15, 28)$. Set $D := 2L - C$. Then $(D^2, L.D) = (0, 1)$, contradicting the global generation of L by [SD].

If $m_x = 5$, the only solutions to (6) are $(L.C, C^2) = (12, 18)$, $(13, 18)$ and $(13, 20)$. Set $D := C - L$. Then $(D^2, L.D) = (2, 4)$, $(0, 5)$ and $(2, 5)$, respectively. But then any curve $D_0 \in |D|$ with a singular point y satisfies $\varepsilon_{D_0, y} \leq \frac{L.D}{2} < \frac{L.C}{5} = \varepsilon_{C, x} = \varepsilon(L)$, a contradiction.

If $m_x = 4$, the only solutions to (6) are $(L.C, C^2) = (9, 10)$, $(10, 10)$ and $(10, 12)$. As above, set $D := C - L$. Then $(D^2, L.D) = (0, 1)$, $(-2, 2)$ and $(0, 2)$, respectively. The first and third case contradict global generation and very ampleness, respectively, of L by [SD]. The middle case yields the contradiction $\varepsilon(L) \leq L.D = 2 < \frac{5}{2} = \varepsilon_{C, x}$.

If $m_x = 3$, the only solutions to (6) are $(L.C, C^2) = (6, 4)$, $(7, 4)$ and $(7, 6)$. Then $D := L - C$ satisfies $(D^2, L.D) = (0, 2)$, $(-2, 1)$ and $(0, 1)$, respectively, yielding the same contradictions as in the previous case.

If $m_x = 2$ and $C^2 \geq 2$, the only solutions to (6) are $(L.C, C^2) = (4, 2)$ and $(5, 2)$. In the first case the Hodge index theorem yields $L \sim 2C$, contradicting the very ampleness of L by [SD]. In the second case $D := L - C$ satisfies $(D^2, L.D) = (0, 3)$ contradicting the fact that the homogeneous ideal of X is generated only by quadrics by [SD].

If $m_x = 2$ and $C^2 = 0$, then by (6) we must have $L.C \leq 5$. Since L is very ample and the homogeneous ideal of X is generated only by quadrics, we must have $L.C \geq 4$ by [SD]. Finally, if $m_x = 1$, then (6) yields $L.C \leq 2$, whence $C^2 = -2$ by [SD] as L is very ample and C is irreducible. Thus $C \cong \mathbb{P}^1$. As X is assumed not to contain lines, we must have $L.C = 2$.

To summarize, we have proved that if $\varepsilon(L) < \frac{8}{3}$, then the only possible irreducible Seshadri curves satisfy $(L.C, C^2, m_x) = (5, 0, 2)$, $(4, 0, 2)$, $(2, -2, 1)$.

In any of the above cases, the divisors L and C are linearly independent in $\text{Pic } X$. There are complete intersections X of type $(2, 2, 2)$ in \mathbb{P}^5 such that $\text{Pic } X \cong \mathbb{Z}[L] \oplus \mathbb{Z}[C]$, with $L = \mathcal{O}_X(1)$ and C as in any of the above cases, cf. [Kn2, Thms. 1.1 and 6.1]. One can verify that in each of the cases there are no divisors with the intersection properties of the two other cases, whence $\varepsilon(X) = \varepsilon_{C, x}$. Therefore, cases (b-ii) and (b-iii) do occur, and they occur in codimension one subsets of the space of all such surfaces by [Ko, Thm. 14] or [G-H2, p. 594]. This finishes the proof of the theorem. \square

4. CURVES WITH POINTS OF MAXIMAL EXPECTED MULTIPLICITIES

In a nonempty linear system $|D|$ on a smooth projective surface X one expects the existence of curves with a point of multiplicity m for any m satisfying

$$(7) \quad \dim |D| - \frac{1}{2}m(m+1) + 2 \geq 0.$$

We therefore make the following definition.

Definition 4.1. A curve C in a linear system $|D|$ on a smooth projective surface X is said to have a point of *maximal expected multiplicity* if it has a point of multiplicity m_0 , where m_0 is the maximal integer m satisfying (7).

We note that the curves with a triple point constructed in [Ga] and in Proposition 2.2 that we used in the proof of Theorem 1.2 are curves in the hyperplane system $|L|$ with a point of maximal expected multiplicity. The reason why we were able to compute all Seshadri constants was the fact that these curves induced a Seshadri constant $< \sqrt{L^2}$. Any lower Seshadri constant must then be induced by an irreducible Seshadri curve and the rest of the proof is merely to find out the different intersection properties of these. The same approach would of course also work if the curves with a point of maximal expected multiplicity were elements of $|nL|$ for $n > 1$, and also if the maximal expected multiplicity is > 3 . The argument in [Ga] to prove the existence of curves with a triple point can in principle be generalized to curves with singularities of multiplicities $m > 3$. One may therefore ask how far this procedure will reach if one generalizes [Ga]. The next proposition shows, however, that there is little to win: one can at the best use this procedure to compute Seshadri constants on $K3$ surfaces of degrees 14 and 24.

Proposition 4.2. *Let L be a globally generated line bundle on a $K3$ surface X such that $L^2 \geq 4$. Assume that there is a curve $C \in |nL|$, for some $n \in \mathbb{Z}^+$, with a point x of maximal expected multiplicity m such that $\varepsilon_{C,x} < \sqrt{L^2}$. Then*

$$(L^2, n, m) \in \{(6, 1, 3), (6, 2, 5), (8, 1, 3), (14, 1, 4), (24, 1, 5)\}.$$

Proof. From (7) we have

$$\frac{1}{2}m(m+1) - 2 \leq \dim |nL| = \frac{1}{2}n^2L^2 + 1.$$

By assumption, $\varepsilon_{C,x} = \frac{L \cdot C}{m} = \frac{nL^2}{m} < \sqrt{L^2}$, whence $n^2L^2 < m^2$, so that $m \leq n^2L^2 - m^2 + 6 \leq 5$. As $L^2 \geq 4$, the only solutions are the ones listed. \square

We remark that also in the case of quartic surfaces [Ba1], curves with a point of maximal expected multiplicity are not enough to compute all Seshadri constants: in fact, in the case $\varepsilon(L) = \frac{4}{3}$ the irreducible Seshadri curves are rational curves $C \in |L|$ with a triple point, whereas the maximal expected multiplicity is 2. However, this does not happen on the general quartic, so hyperplane sections with a point of maximal expected multiplicity 2 are Seshadri curves yielding $\varepsilon(X) = 2$ in the general case.

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